INTEGRATED VEHICLE ROUTING AND COLLECTION POINT OPTIMIZATION FOR URBAN WASTE MANAGEMENT: A LAGRANGIAN RELAXATION APPROACH WITH WALKING-DISTANCE CONSTRAINTS

K. Worasan[1] and K. Moonsri[1*]

[1]Faculty of Business Administration and Accountancy, Khon Kaen University, Khon Kaen, Thailand

*Corresponding Author's Email: karnmo@kku.ac.th

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ABSTRACT: Thailand's urban areas face significant waste management challenges due to increasing population density and inefficient collection systems. This results in high operational costs and significant environmental impact. Moreover, traditional waste management methods lack integrated optimizations of collection point placement and vehicle routing, leading to suboptimal resource utilization. This study develops an integrated optimization approach that simultaneously determines optimal collection point locations and vehicle routes while minimizing fuel consumption and ensuring resident accessibility through walking-distance constraints. A vehicle routing problem with generalized clustering and maximum walking distance (VRPGC-MWD) is formulated in this study as a mixed-integer linear programming model that integrates collection point selection with vehicle routing optimization. A Lagrangian relaxation (LR) approach is proposed and tested on instances ranging from 7 to 30 locations, with 2–5 vehicles and 3–8 collection points, to efficiently solve this Non-deterministic Polynomial-time hard problems. The LR approach maintains optimality gaps below 7.73% for small instances. For larger instances where exact methods fail, the LR approach offers efficient solutions with gaps between 8.91% and 13.10%. This research contributes to the first integrated model combining collection point selection with vehicle routing under walking-distance constraints, offering Thai municipalities a practical tool to support waste management.".

KEYWORDS: *Vehicle Routing, Walking Distance, Waste Collection, Lagrangian*

Relaxation

1.0 INTRODUCTION

Thailand's manufacturing sector faces mounting waste management challenges that directly impact production efficiency and increase supply chain costs. Industrial zones generate substantial waste, requiring optimized coordinated collection systems aligned with manufacturing schedules to prevent production disruptions [1]. Traditional waste collection methods create operational bottlenecks that increase costs and reduce manufacturing competitiveness.

Thailand's urban–industrial landscape presents unique optimization challenges, such as high-density manufacturing areas with complex routing requirements, variable waste generation patterns across production cycles, and the need for strategic collection point placement that balances worker accessibility with vehicle efficiency. Current approaches fail to integrate collection point selection with vehicle routing—a critical gap in manufacturing environments, in which operational efficiency directly affects production costs. Waste collection optimization has advanced through generalized vehicle routing problems (GVRP), yet existing studies within operations research treat routing and collection point selection separately despite their interdependence in manufacturing logistics. Current Lagrangian relaxation applications inadequately handle the combination of clustering decisions and accessibility constraints essential for manufacturing-integrated waste collection systems.

Recent waste collection optimization studies have revealed fundamental methodological inconsistencies and conflicting priorities across three main areas. Methodological approaches demonstrate contradictory complexity–practicality trade-offs. Rahmanifar et al. [2] and Hà et al. [3] pursue sophisticated temporal synchronization and hybrid exact-metaheuristic approaches, although these methods paradoxically increase computational complexity without addressing the basic walking-distance constraints that must be met to ensure accessibility in waste collection. Conversely, despite municipal waste patterns being sufficiently predictable, Biesinger et al. [4] advocate for stochastic demand modeling to question whether such complexity is justified.

Limitations in application scope have created significant research gaps. Vehicle routing studies by Li et al. [5], Buhrkal et al. [6], and Nirwan et al. [7] focus on specialized roll-on/roll-off operations, general VRP applications, and middle-mile delivery logistics, respectively. These studies do not adequately address the specific clustering requirements of municipal waste-collection systems. Similarly, Crainic et al. [8] examine freight delivery applications but fail to translate their findings to municipal service contexts in which accessibility priorities are not aligned with freight efficiency objectives.

Technology integration research has exhibited disconnects between implementation and theory. Liu et al. [9], Son [10], Cheng et al. [11], Idrissi et al. [12], and Giasoumi et al. [13] demonstrate the technological benefits of vehicle routing, including data-driven optimization, GIS integration, and and IoT-based solutions. However, these approaches systematically overlook implementation costs and technical complexity barriers that challenge resource-constrained municipalities, suggesting that these academic advancements cannot meet the constraints necessary for practical adoption.

Specialized approaches to waste management have not had their applicability broadly validated. Masmoudi et al. [14] address multi-compartment vehicles, Erdem [15] focuses on electric vehicle applications, Ghiani et al. [16] optimize transfer stations, Wang et al. [17] consider passenger walking distances, and Rattanawai et al. [18] examine mountainous terrain with infectious waste. While each study contributes to understanding specific contexts, the collective research landscape lacks the comprehensive integration necessary for standard urban waste collection scenarios. The persistent treatment of collection point selection and vehicle routing as separate optimization problems, despite their interdependence in determining overall system efficiency, is the critical limitation shared across all approaches. This is particularly relevant for manufacturing-integrated waste management, as operational coordination directly affects production costs.

This research advances the field by determining the number of collection points within municipal districts while simultaneously optimizing vehicle routing decisions. When combined with practical constraints such as walking distances, vehicle capacities, and fuel consumption optimization, this integration provides a comprehensive solution to urban waste management challenges that existing fragmented approaches fail to address adequately.

This paper presents a mixed integer linear programming (MILP) model for optimizing waste collection in Thailand's urban environment. The model simultaneously addresses two critical aspects of Thailand's waste management: the strategic location of collection points and efficient vehicle routing. This integrated approach serves Thai municipalities seeking to modernize their waste management systems within their existing infrastructure and resource constraints.

The MILP model incorporates essential practical considerations specific to the Thai context: maximum walking distances for residents, vehicle capacity constraints, and route duration limits. The objective function focuses on minimizing fuel consumption, aligning with environmental sustainability goals and municipal economic interests.
This research contributes to the field through several advances. The comprehensive MILP model specifically addresses Thailand's urban waste collection challenges through integrated collection point selection and vehicle routing optimization. The incorporation of practical constraints reflects actual operational conditions, while fuel consumption minimization addresses environmental sustainability concerns.

The remainder of this paper is organized as follows: Section 2 presents the mathematical formulation of the MILP model, Section 3 describes the Lagrangian relaxation solution methodology, Section 4 presents computational results and analysis, and Section 5 concludes with recommendations for practical implementation and future research directions.

2.0 MATHEMATICAL FORMULATION

This section presents a novel mathematical formulation that addresses the critical gaps in existing waste collection optimization research. Unlike previous approaches, which treat collection point selection and vehicle routing separately, this integrated model simultaneously optimizes both decisions to achieve superior solutions. The key innovation bridges three fundamental gaps: (1) integrating clustering decisions with routing optimization, (2) incorporating walking-distance constraints for urban accessibility, and (3) determining optimal collection point quantities rather than assuming fixed points.

This formulation advances beyond existing GVRP and WCVRP models by capturing interdependencies between collection point placement and vehicle routing efficiency; a relationship that previous fragmented approaches have failed to optimize effectively.

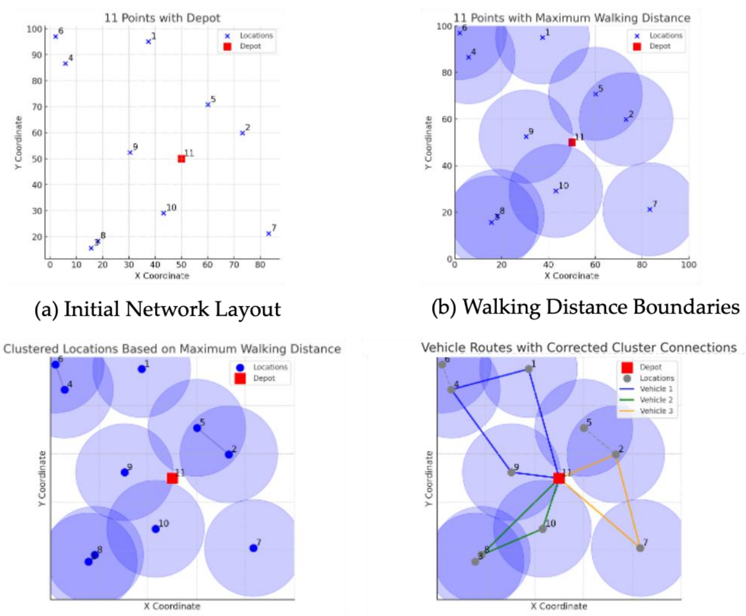
The model integrates vehicle routing with a cluster design for waste collection optimization, incorporating walking distance constraints, vehicle capacities, and operational requirements specific to urban waste management optimization.

2.1 Problem Statement

The VRPGC-MWD integrates vehicle routing optimization with walking-distance constraints, vehicle capacity, and fuel consumption. This study models waste collection planning through strategic cluster formation and multivehicle route optimization.

The VRPGC-MWD model presented in this study differs fundamentally from previous waste collection optimization approaches by simultaneously addressing both strategic and operational decisions. While existing models typically conceive of collection point selection and route optimization as separate problems, this integrated approach captures the interdependencies between these decisions. This integration is particularly crucial in Thailand's urban context, in which the complex interplay between walking accessibility, vehicle routing constraints, and waste generation patterns creates unique optimization challenges that have not been adequately addressed in the literature.

The problem utilizes an 11-node network (see Figure 1) configuration in which a central depot serves ten service locations. Maximum walking-distance constraints create overlapping service regions, whereas multiple vehicles with varying capacities serve the clustered locations. The developed model simultaneously optimizes cluster formation, route design, workload distribution, and service coverage.



(a) Initial Network Layout (b) Walking Distance Boundaries

(c) Location Clustering (d) Final Vehicle Routes

Figure 1: Solution Framework of VRPGC-MWD: (a) Initial network layout of 11 points, including the depot; (b) walking-distance boundaries illustrating feasible service areas; (c) resulting location clusters based on maximum walking distance; (d) final vehicle routes with corrected cluster connections.

Figure 1 illustrates the solution development process of the VRPGC-MWD model. Figure 1(a) shows the spatial distribution of the service locations and the depot, while Figure 1(b) details the maximum walking-distance constraints through overlapping circles to define feasible service regions. Figure 1(c) depicts the resulting location clusters based on walking-distance constraints. Lastly, Figure 1(d) presents the optimized solution with vehicle-specific routes shown in different colors, with the solid lines representing vehicle paths within the established clusters.

The mathematical formulation of the VRPGC-MWD requires the following sets, parameters, and decision variables:

Table 1: Sets, Parameters, and Decision Variables

Type	Definition
Sets	
V	Set of vehicles
N	Set of locations
D	Set of depots
P	Number of collection points to select
Parameters	
Q_v	Capacity of vehicle $v \in V$
W_i	Waste generation at location $i \in N$
D_{max}	Maximum walking distance
$Dist_{ij}$	Distance between location $i, j \in N$
T_i	Service time at location $i \in N$
T_{max}	Maximum route duration
M	Large constant for subtour elimination
F_v	Fuel consumption per unit distance $v \in V$
$Speed_v$	Average speed of vehicle $v \in V$
Decision Variables	
x_{vij}	$Binary\ variable{:}\ 1\ if\ vehicle\ v \in V\ travels\ from\ i \in N\ to\ j \in N$
u_{vi}	$Auxiliary\ variable\ for\ subtour\ elimination\ for\ vehicle\ v \in V, i \in N$
y_i	$Binary\ variable{:}\ 1\ if\ location\ i \in N\ is\ selected\ as\ a\ collection\ point$
z_{ij}	$Binary\ variable{:}\ 1\ if\ waste\ from\ location\ \in N$ $is\ assigned\ to\ collection\ point\ j \in N$
G_j	$Total\ waste\ collected\ at\ collection\ point\ j \in N$
C_{vj}	$Waste\ collected\ by\ vehicle\ v \in V\ at\ location\ j \in N$

The following section presents the complete mathematical formulation of the VRPGC-MWD model. The objective function and all associated constraints are detailed in Constraints (1)–(14), as shown below.
The VRPGC-MWD model consists of an objective function and 14 key constraints. The objective function (1) minimizes the total travel distance and fuel consumption across all vehicle routes.

The objective is defined as follows:

$$\sum_{v \in V} \sum_{i \in N} \sum_{j \in N, i \neq j} Dist_{ij}\, x_{vij} F_v \tag{1}$$

Constraints
The constraints ensure operational feasibility through several mechanisms. Constraints (2)–(4) manage the selection and assignment of collection points, ensuring that exactly P points are chosen and that each location is properly assigned.

$$\sum_{i \in N, i \notin D} y_i = P \tag{2}$$

$$\sum_{j \in N, j \notin D} z_{ij} = 1, \quad \forall i \in N, i \notin D \tag{3}$$

$$z_{ij} \leq y_j, \quad \forall i, j \in N, i \notin D, j \notin D \tag{4}$$

Walking-distance requirements are enforced by constraint (5), limiting the maximum distance between locations and their assigned collection points. Waste collected accumulation at collection points is controlled by constraint (6), while constraint (7) ensures that vehicle routes have their flow conserved.

$$Dist_{ij} z_{ij} \leq D_{max}, \quad \forall i, j \in N, i \notin D, j \notin D \tag{5}$$

$$G_j = W_j y_j + \sum_{i \in N, i \neq j, i \notin D} W_i z_{ij}, \quad \forall j \in N, j \notin D \tag{6}$$

$$\sum_{i \in N, i \neq j} x_{vij} = \sum_{i \in N, i \neq j} x_{vji}, \quad \forall v \in V, j \in N \tag{7}$$

Depot operations are governed by constraints (8)–(9). This ensures that all vehicles start and end their routes at the depot. Vehicle capacity and service requirements are addressed through constraints (10)–(11), which link waste collection to route decisions.

$$\sum_{j \in N, j \notin D} x_{v,first(D),j} \leq 1, \quad \forall v \in V \tag{8}$$

$$\sum_{i \in N, i \notin D} x_{vi,first(D)} \leq 1, \quad \forall v \in V \tag{9}$$

$$\sum_{v \in V} C_{vj} = G_j, \quad \forall j \in N, j \notin D \tag{10}$$

$$\sum_{j \in N, j \notin D} C_{vj} \leq Q_v, \quad \forall v \in V \tag{11}$$

Constraint (12) prevents subtours in vehicle routes. Constraint (13) enforces route duration limits, ensuring that operational hours are respected. Finally, constraint (14) enforces non-negativity for vehicle flow, waste amount, and collection capacity variables across all vehicles and locations.

$$u_{vi} - u_{vj} + M x_{vij} \leq M - 1, \quad \forall v \in V, i, j \in N, i \neq j, i \notin D, j \notin D \tag{12}$$

$$\sum_{i \in N} \sum_{j \in N, i \neq j} \left(\left(Dist_{ij} / Speed_v \right) + T_j \right) x_{vij} \leq T_{max}, \quad \forall v \in V \tag{13}$$

$$u_{vi}, G_j, C_{vj} \geq 0, \quad \forall v \in V, i \in N, j \in N. \tag{14}$$

3.0 SOLUTION METHODOLOGY

This section presents a Lagrangian relaxation (LR) approach for solving the VRPGC-MWD. In contrast to general LR implementations, this LR approach specifically targets the hierarchical relationship between strategic collection point selection and tactical routing decisions, allowing for efficient decomposition while maintaining solution quality. The proposed methodology addresses key research questions regarding the effectiveness of constraint relaxation and computational tractability for solving practical problems in Thai municipal waste management systems.

3.1 Lagrangian Relaxation Framework

According to Song et al. [19], the computational complexity of VRPGC-MWD relaxes the linking constraints between routing and collection point selection. Let λ_j be the Lagrangian multipliers associated with cluster-formation constraints and μ_v be the multipliers for vehicle-capacity constraints. The relaxed objective function becomes:

$$\begin{aligned} LR(\lambda, \mu) = & \sum_{v \in V} \sum_{i \in N} \sum_{j \in N} Dist_{ij} \cdot x_{vij} \cdot F_v \\ & + \sum_{j \in N} \lambda_j \cdot \left(\sum_{v \in V} \sum_{i \in N} x_{vij} - M \cdot y_j \right) \\ & + \sum_{v \in V} \mu_v . \left(\sum_{j \in N \setminus \{depot\}} C_{vj} - Q_v \right) \end{aligned} \tag{15}$$

The Lagrangian relaxation objective function (15) incorporates three key components: the original objective minimizing travel and fuel costs, relaxed linking constraints between routes and collection points weighted by multiplier λ_j, and relaxed vehicle capacity constraints weighted by multiplier μ_v.

3.2 Subproblem Decomposition

This study decomposes the relaxation function into two subproblems:

Routing subproblem (SP1):

The routing subproblem (16) optimizes vehicle paths with modified costs from Lagrangian multipliers.

$$\sum_{v \in V} \sum_{i \in N} \sum_{j \in N} \left(Dist_{ij} + \lambda_j\right) \cdot x_{vij} \cdot F + \sum_{v \in V} \mu_v \cdot routeLoad_v \tag{16}$$

Collection point selection subproblem (SP2):

The collection point selection subproblem (17) determines the optimal location assignments by considering penalty terms from the relaxation function.

$$-\sum_{j \in N} M \cdot \lambda_j \cdot y_j \tag{17}$$

3.3 Subgradient Optimization

Multiplier updates:

Multiplier updates are governed by equations (18)–(19). The λ_j multiplier update (18) resolves to violations of linking constraints between routes and collection points. The μ_v multiplier update (19) addresses violations of vehicle capacity constraints. These updates use step size s_t (20), which is calculated using parameter α, the current optimality gap, and subgradient information.

$$\lambda_j^{t+1} = \left(0, \lambda_j^t + s_t \cdot \left(\sum_{v \in V} \sum_{i \in N} x_{vij} - M \cdot y_j\right)\right) \tag{18}$$

$$\mu_v^{t+1} = \left(0, \mu_v^t + s_t \cdot (routeLoad_v - Q_v)\right)] \tag{19}$$

Step size calculation:

$$s_t = \alpha \cdot \frac{UB - LB}{\left||subgradient|\right|^2} \tag{20}$$

4.0 RESULTS

The computational experiments evaluate the performance of the proposed Lagrangian relaxation approach across multiple problem instances, examining solution quality, computational efficiency, and scalability characteristics for urban waste collection applications.

4.1 Implementation and Convergence

The duality gap is calculated as follows:

$$Gap = \frac{UB - LB}{LB} \times 100\% \tag{21}$$

Solution quality is evaluated through duality gaps comparing the best feasible solutions (UB) against Lagrangian bounds (LB). Both the mathematical model and the LR algorithm were implemented in IBM ILOG CPLEX 22.1.1 with the following LR parameters: initial step size $\alpha > 0$ [19], maximum iterations = 200 [20], and convergence tolerance = 0.001 [19]. The tests were ran on a platform with an Intel Core i7 (2.6 GHz) CPU and 16 gigabytes of RAM.

4.2 Computational Performance and Solution Quality Analysis

Table 2 compares the results from both methods across 12 instances with varying locations (L), vehicles (V), and collection points (G). For smaller instances ($L \leq 15$), both methods obtain solutions, with LR demonstrating gaps below 8%. Larger instances ($L \geq 20$) highlight LR's advantage over the mathematical model, which fails to find optimal solutions within 3,600 seconds. Conversely, LR efficiently provides solutions. Figure 2 illustrates this comparison: The blue line shows optimal values, while the red dashed line represents LR solutions. The shaded area between these lines indicates optimality gaps and demonstrates LR's effectiveness for larger problem instances in which exact methods become computationally prohibitive.

Table 2: Comparison of Optimal Solutions

Instance Size (L/V/G)	Optimal Value (time(s))	LR (time(s))	Gap (%)
7/2/3	340(0.59s)	328(0.02s)	3.66
7/2/4	370(0.45s)	355(0.02s)	4.23
11/2/3	310(1.31s)	295(0.03s)	5.08
11/2/4	310(1.25s)	292(0.05s)	6.16
15/3/4	340(50.41s)	318(0.05s)	6.92
15/3/5	*390	362(0.03s)	7.73
20/3/5	*550	505(0.16s)	8.91
20/3/6	*580	528(0.22s)	9.85
25/4/6	*600	542(0.25s)	10.70
25/4/7	*660	592(0.19s)	11.49
30/5/7	*690	614(0.33s)	12.38
30/5/8	*760	672(0.27s)	13.10

Note: *Unable to find optimal solution within time limit (3,600 seconds)

Figure 2 demonstrates the solution quality comparison between the optimal and LR approaches, with the blue line representing optimal values and the red dashed line showing LR solutions. The shaded gap area illustrates LR's effectiveness when exact methods become computationally intractable. This pattern supports Hà et al.'s [3] findings on exact algorithm limitations for larger vehicle routing problems. The predictable gap progression aligns with Song et al.'s [19] work, which finds that Lagrangian relaxation methods provide stable solution quality and address municipal waste collection scalability requirements where exact methods prove inadequate.

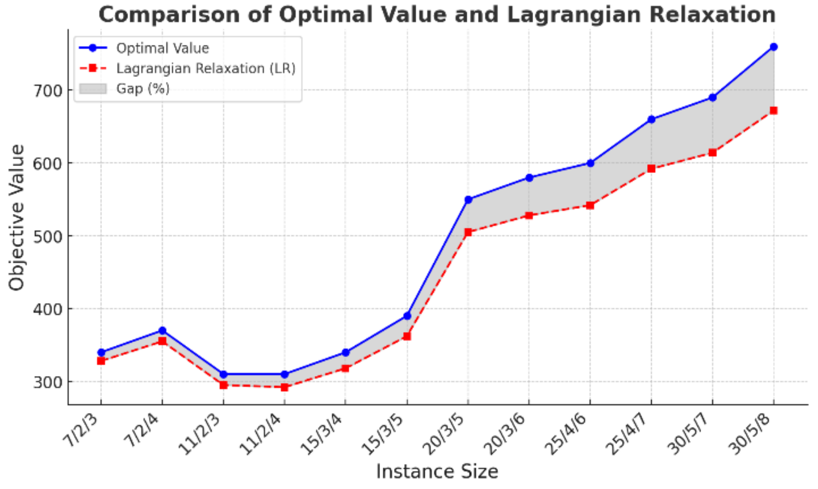


Figure 2: Gap Analysis between Optimal Value and Lagrangian Relaxation

The computational performance comparison presented in Figure 3 reveals a dramatic contrast between the solution approaches. The exact method exhibits exponential growth in computational time, failing to terminate within the 3,600-second limit for instances larger than 15/3/5. In stark contrast, the LR approach maintains consistently low computational times below 0.5 seconds even for the largest instances (30/5/8), demonstrating its superior scalability for practical applications.

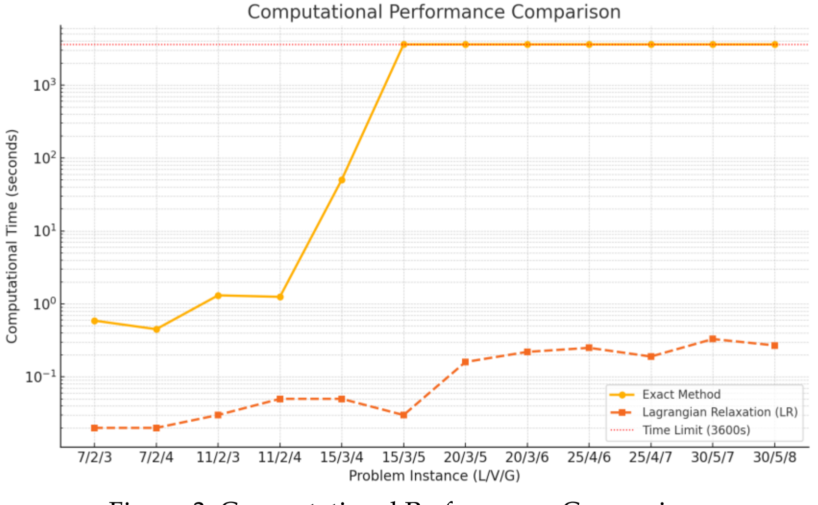


Figure 3: Computational Performance Comparison

4.3 Solution Quality Degradation and Parameter Sensitivity

Figure 4 presents a critical analysis of the relationship between problem size and solution quality, revealing a predictable degradation pattern following the trend line y = 0.38x + 1.43. This systematic increase in optimality gaps from 3.66% for small instances (7/2/3) to 13.10% for large instances (30/5/8) provides decision-makers with clear expectations regarding the trade-offs between computational efficiency and optimal solutions. The linear degradation pattern supports Song et al.'s [19] findings: Lagrangian relaxation methods provide consistent performance characteristics. This contrasts Hà et al.'s [3] observations of variable performance in exact-metaheuristic combinations as well as Biesinger et al.'s [4] stochastic approaches that produce inherently variable results. This predictable relationship enables municipal planners to anticipate when their solutions degrade in quality and make informed decisions when considering problem scale against computational requirements.

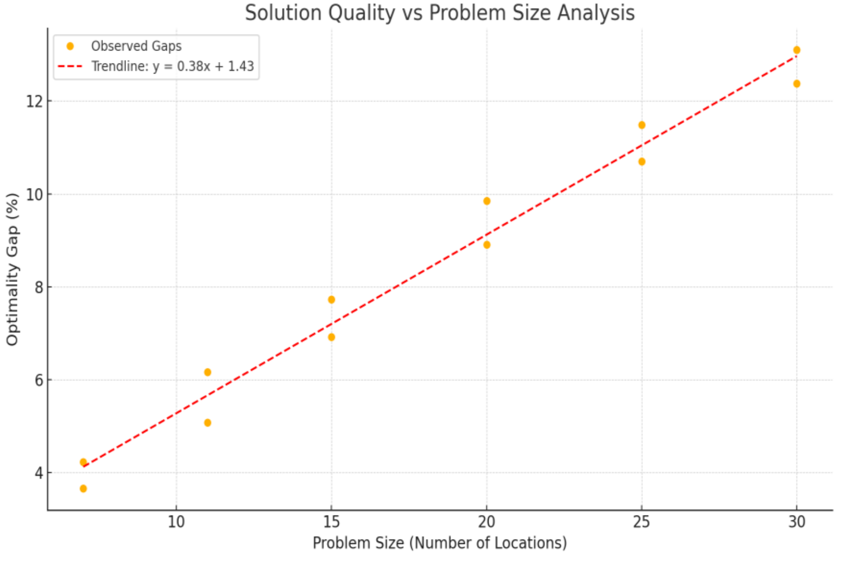


Figure 4: Solution Quality Versus Problem Size Analysis

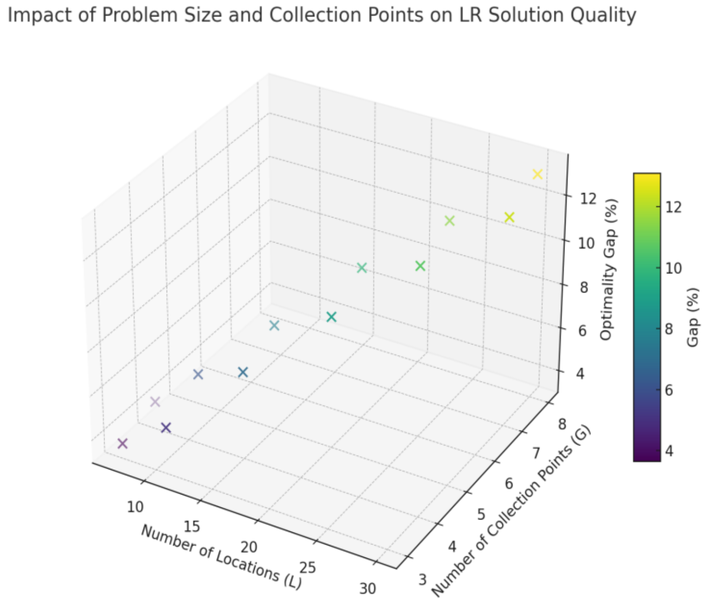


Figure 5: Impact of Problem Size and Collection Points on LR Solution Quality (3D Visualization)

The three-dimensional analysis in Figure 5 examines the combined impact of problem size and collection point configuration on LR solution quality, indicating that while increasing the number of collection points generally improves solution quality, this effect diminishes for larger problem instances—a result that aligns with Wang et al.'s [17] findings on reduced marginal benefits as system complexity increases. The color gradient indicates that optimal parameter combinations exist for different problem scales, suggesting the need for adaptive parameter selection strategies that contrast with the fixed parameter approaches proposed by Buhrkal et al. [6] and uniform optimization strategies reported by Masmoudi et al. [14]. This also demonstrates that scale-sensitive parameter adaptation, rather than one-size-fits-all approaches, is necessary for integrated clustering-routing problems.

5.0 CONCLUSION

This paper introduces a vehicle routing problem with generalized clustering and maximum walking distance (VRPGC-MWD) that addresses waste collection optimization challenges in Thailand's urban environment. The mathematical model integrates collection point selection with vehicle routing decisions, representing a significant advancement over previous approaches that treated these aspects separately.

The proposed Lagrangian relaxation approach demonstrates strong performance across varying problem sizes, with optimality gaps ranging from 3.66% to 13.10%. For smaller instances (≤15 locations), the LR method maintains gaps below 7.73% while significantly reducing computational time compared to exact methods. As problem size increases to medium-scale instances (20 locations), gaps increase moderately to 8.91-9.85%, remaining within acceptable limits for practical applications. The LR approach provides high-quality solutions with gaps between 10.70% and 13.10% For larger problems representing real municipal districts (25–30 locations) in which exact methods fail to find optimal solutions within reasonable time limits.

The relationship between problem size and solution quality follows a predictable pattern: Optimality gaps increase systematically from the smallest instance (7/2/3) to the largest (30/5/8). This predictability provides valuable implementation insights that allow municipal planners to anticipate solution quality for different problem scales.

Future research should explore comparative analysis against metaheuristic approaches, model extensions incorporating dynamic waste generation patterns and time-dependent travel speeds, and examine hybrid approaches combining LR with metaheuristics. This model could be adapted for use in recyclable material collection and enhanced through IoT integration for dynamic routing strategies.

AUTHOR CONTRIBUTIONS

K. Moonsri and K. Worasan: Conceptualization, Methodology, Software, Writing-Original Draft Preparation; K. Moonsri and K. Worasan: Data Curation, Validation, Supervision; K. Moonsri and K. Worasan : Software, Validation, Writing-Reviewing and Editing.

CONFLICTS OF INTEREST

The manuscript has not been published elsewhere and is not under consideration by other journals. All authors have approved the review, agree with its submission and declare no conflict of interest on the manuscript.

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