PROPERTIES OF OPTIMAL SOLUTION OF INDEFINITE MATRIX CONSTRAINT IN LINEAR PROGRAMMING

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ABSTRACT: This study investigated characteristics of indefinite random square matrices which represented the constraints of Linear programming problems. MATLAB simulation was used to generate different size of indefinite random non-symmetric square matrices. Solutions of primal problem and dual problem were deliberated and discussed. Based on simulation results, duality gap found in some of the indefinite non-symmetric matrices and those matrices could not obtain optimal solution whereas some ID matrices that fulfill certain conditions could achieve optimal solution and no duality gap is found. An indefinite non-symmetric matrix with all positive off-diagonal entries and alternate signs of determinant of leading principal minors surely confirmed the existence of optimal solution in linear programming problems.

KEYWORDS: *Random square matrices, linear programming, indefinite, duality gap*

1.0 INTRODUCTION

Linear programming (LP) is one of the most active studies on optimization problems [1]. In terms of LP problems, the duality solution are always taken into consideration to validate the accurateness of the solution [2]. LP problem is known as primal problem while its reflection is referred as dual problem [3]. The property of primal and dual provide deeper and further valuable understanding of the optimal solution to LP problems [4].

Primal LP problem can be shown as [5]:

Min $$f = c^T x \quad (1)$$

Subject to: $$A_{nxn}x \leq b \quad (2)$$

$$x \geq 0 \quad (3)$$

Dual LP Problem can be written as:

Max	$v = b^T y$	(4)
Subject to:	$A^T_{nxn} y \geq c$	(5)
	$y \geq 0$	(6)

where $A_{nxn} = (a_{ij})$ is a matrix, $c = (c_1, c_2, \cdots, c_n)$; $b = (b_1, b_2, \cdots, b_n)$; $x = (x_1, x_2, \cdots, x_n)$; $y = (y_1, y_2, \cdots, y_n)$ are column vectors. Every elements (*a, b* and *c*) is ∈ R.

The concept of duality in LP has been widely discussed. One of the characteristics of duality, i.e. strong duality between the lower and upper bound formulation of shakedown analysis of framed structure has been proven [6]. Besides, weak duality theory states that if primal is unbounded then dual is infeasible [7].

The indefinite (ID) linear programs are complicated and harder than the definite and semi-definite linear programs. A lot of studies focus on the symmetric ID linear systems, i.e. study on the effect of a modified positive or negative-stable splitting method to the complex symmetric indefinite linear programs [8] and duality gap can exist in the primal and dual semi-indefinite linear programs [9].

Researchers may face difficulties in solving ID matrices. Lee and Zhang proved that the proposed algorithms can only improve the accuracy and stability of the solutions and cannot guarantee the achievement of optimal solution to the linear problems [10]. Other studies focus on the practical usage of the ID matrix in solving linear problems rather than theoretical development. There are limited sources on the effect of characteristics of ID matrix to the linear solutions.

Xu [11] presented the eigenvalue bounds of two classes of two-by-two block indefinite matrices to solve problems. The effect of symmetric matrices to the linear solutions has been discussed by Romli et al.[12] by showing that indefinite symmetric matrices might or might not provide solution to the linear problems and it does not guarantee the existence of optimal solution.

Hence, this paper extends the research of [12] by focusing on investigating characteristics of ID non-symmetric square matrices and trying to identify properties of ID matrices which can guarantee the optimal solution to the linear problems. Randomized parameters are strongly suggested in computing applications as they can provide outstanding performance than deterministic algorithms but still giving accurate solutions to the problems [13]. This study focused on small size of LP problems, hence, simplex method was applied due to its efficiency in practice for solving various small types and sizes of LP problems [14]. This algorithm is carried out by moving along the extreme points in the edges of a feasible region until optimal solution is found [15].

2.0 METHODOLOGY

The methodology of this study started with defining the problem statement, objectives and scopes. Then, the coefficients of objectives function, square matrices and right hand side (RHS) vectors were randomly generated by MATLAB. The generated matrix was tested by verifying that its characteristics belonged to indefinite (ID) matrix. Once the parameters were obtained, the LP problems were ready to solve.

2.1 Generate Random Non-symmetric Square Matrices

Create square matrices using random generator. For example,

$$A_{nxn} = \begin{bmatrix} a_{11} & a_{12} & \dots a_{1n} \\ a_{12} & a_{22} & \dots a_{2n} \\ & & \vdots \\ a_{1n} & a_{2n} & \dots a_{nn} \end{bmatrix} \tag{7}$$

This paper aimed at identifying further information on the effect of ID matrix to the LP solutions and the dimension of the generated ID non-symmetric square matrices was up to 20x20, which is 5x5, 10x10 and 20x20.

2.2 Verify Properties of ID Random Non-symmetric Square Matrices

First step of verification was determined through the determinant of leading principal minors. The determinant of any square matrix A_{nxn} was denoted as $|A_{nxn}|$ or $\det(A_{nxn})$. Let α_k be the determinant of the leading principal minor of order k of A_{nxn}, as shown:

$$\alpha_1 = a_{11}\,, \alpha_2 = \ det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \alpha_n = det\, A_{nxn} \tag{8}$$

The matrix was ID matrix if and only if it failed the definiteness test sign and the semi-definiteness test sign or it was neither definite nor semi-definite matrix. Else, a square matrix A can be classified as indefinite matrix if and only if [16]:

1. $\alpha_k < 0$ for some even k or;
2. $\alpha_{k1} > 0$ for odd k_1 and $\alpha_{k2} < 0$ for even k_2.

Then, quadratic form test $f(x) = x^T A_{nxn} x$ was required, where x was any vector. It was applied by means of each of the generated square matrices which were multiplied with vector x. By looking at the sign of the results of the quadratic form test of the real square matrix A_{nxn}, researchers were able to identify the definiteness properties of a matrix.

Last step was to determine the eigenvalues of the generated random square matrix. If the eigenvalues belonged to matrix A consisted of both positive and negative values, then the matrix was defined as ID matrix [17]. For an indefinite matrix, A, let assume λ is an eigenvalue of A, then for any eigenvector x that belong λ, it shows that

$$x^T Ax \;=\; \lambda x^T x \;=\; \lambda||x||^2 \tag{9}$$

Thus,

$$\lambda \;=\; \frac{x^T Ax}{||x||^2} \tag{10}$$

2.3 Solve and Validate the Primal-Dual Solution

Once the parameters were generated, MATLAB was used to solve the LP primal and dual problems. The simulation results were validated by Lingo software. All primal and dual solutions were summarized for further analysis and discussion.

3.0 RESULTS AND DISCUSSION

3.1 Results

The study created 180 simulated LP problems. The indefinite (ID) random non-symmetric square matrices had been generated based on three criteria and the samples of generated matrices of order *5x5* were shown in Table 1.

Table 1: Samples of generated A_{5x5} square matrices

	Characteristics														
	Matrix with different sign of leading principal minors					**Matrix with alternate sign of leading principal minors with all positive sign of off-diagonal entries**					**Matrix with alternate sign of leading principal minors with random sign of off-diagonal entries**				
Generated A_{5x5} Square **Matrices**	10	-15	3	6	-6	29	1	7	9	2	24	10	-2	8	2
	-6	-7	-4	4	2	2	-19	1	7	3	5	-21	-2	15	-10
	18	-6	8	-10	-3	3	2	-28	7	6	-8	13	-13	5	9
	-11	4	8	8	-1	7	3	4	-5	1	-3	7	0	-12	3
	2	9	1	-7	6	5	4	5	10	-5	6	-1	1	6	-9

Three criteria were used to verify the generated random square matrices which belonged to ID matrix. Table 2 shows the results of determinant of leading principal minors of some generated ID matrices.

Table 2: Samples of determinant of principal minor of A_{5x5} square matrices

Characteristics	**Generated Square Matrices**	**Leading Principal Minor**				
		α_1	α_2	α_3	α_4	α_5
Matrix with different sign of leading principal minors	**1st**	10	-160	46	-22150	-100075
	2nd	-14	-44	263	-2922	227841
Matrix with alternate sign of leading principal minors with all positive sign of off-diagonal entries	**1st**	29	-553	15856	-90251	113571
	2nd	17	-202	3077	-57741	1382921
Matrix with alternate sign of leading principal minors with random sign of off-diagonal entries	**1st**	24	-554	8192	-58804	469476
	2nd	8	-190	3214	-43530	665867

All matrices in Table 2 fulfilled the characteristics of the ID matrix. A number of matrices consisted of negative value for even orders while other matrices involved alternate sign of leading principal minors with strictly positive at odd orders and strictly negative at even orders. Those characteristics indicated the generated square matrices failed the definiteness and semi-definiteness test sign.

Based on the results, the quadratic form test of these ID matrices involved both negative and positive numbers. Each square matrix might give positive and negative results of quadratic form test. Besides, the eigenvalues test showed that the eigenvalues for all matrices consisted both positive and negative values, thus, the generated matrices were categorized as ID matrices. Once the matrix was verified as ID matrix, then it was ready to solve.

The results of MATLAB for primal and dual solution are tabulated in Table 3. Similar to the findings in [12], some LP problems converged to optimal solution while some of the LP problems failed to get optimal solution, which indicated the duality gap between the LP solutions.

Table 3: Samples of primal and dual solutions

Characteristics	ID Random Square Matrix	Primal LP Solution for ID Random Non-symmetric Square Matrix					
		x_1	x_2	x_3	x_4	x_5	fx
Matrix with different sign of leading principal minors	1st	**20.9735**	6.5242	0	31.5367	35.6819	5666.386
	2nd	1.0e+016	4.77e+016	0	0	9.62e+016	1.14e+019
	ID Random Square Matrix	Dual LP Solution for ID Random Non-symmetric Square Matrix					
		y_1	y_2	y_3	y_4	y_5	fy
	1st	23.72489	1.9299	0	19.3906	35.3134	5666.386
	2nd	9.8454	0	12.8624	0	17.0124	2444.798
Matrix with alternate sign of leading principal minors with all positive sign of off-diagonal entries	ID Random Square Matrix	Primal LP Solution for ID Random Non-symmetric Square Matrix					
		x_1	x_2	x_3	x_4	x_5	fx
	1st	0	13.6502	2.5653	0.8324	13.9503	1231.953
	2nd	0	1.1498	0.8228	4.3694	0	503.135
	ID Random Square Matrix	Dual LP Solution for ID Random Non-symmetric Square Matrix					
		y_1	y_2	y_3	y_4	y_5	fy
	1st	2.5645	0	2.4504	14.9656	6.1594	1231.953
	2nd	0	7.8861	4.0245	0	3.9880	503.135
Matrix with alternate sign of leading principal minors with random sign of off-diagonal entries	ID Random Square Matrix	Primal LP Solution for ID Random Non-symmetric Square Matrix					
		x_1	x_2	x_3	x_4	x_5	fx
	1st	0	0.56e+016	6.10e+016	0.58e+016	1.0e+016	1.93e+018
	2nd	0	0	0	13.4651	33.88372	4282.698
	ID Random Square Matrix	Dual LP Solution for ID Random Non-symmetric Square Matrix					
		y_1	y_2	y_3	y_4	y_5	fy
	1st	6.5625	0	0	2.6917	0.4667	418.638
	2nd	60.3488	0	0	27.6047	0	4282.698

As shown in Table 3, some variables like x and y produced some values while others are 0, but this did not affect the simulation results of LP problems. Some ID matrices in the first and third characteristic could not achieve optimal solution. The objective solution $f(x)$ of these ID matrices involved inconsistency whereby the primal problem was unbounded and the constraints were not restrictive enough while no feasible starting point was found when computing the dual solution.

For the second characteristic, the objective solution $f(x)$ of the primal and dual displayed similar results. All matrices found the optimal solution and no duality gap existed. To validate the simulation results of MATLAB, the generated data in MATLAB were transferred to Lingo Software. Same procedures were replicated in Lingo software to attain the solution for primal and dual problems. Table 4 presents the results acquired by MATLAB and Lingo Software.

Table 4: Validation of MATLAB results by Lingo software

Characteristics	ID Random Square Matrix	Primal in MATLAB	Primal in Lingo	Dual in MATLAB	Dual in Lingo
Matrix with different sign of leading principal minors	1st	5666.386	5666.386	5666.386	5666.386
	2nd	1.139e+019	1.0e+030	2444.798	213.867
Matrix with alternate sign of leading principal minors with all positive sign of off-diagonal entries	1st	1231.953	1231.953	1231.953	1231.953
	2nd	503.135	503.135	503.135	503.135
Matrix with alternate sign of leading principal minors with random sign of off-diagonal entries	1st	1.93e+018	1.0e+030	418.638	37.728
	2nd	4282.698	4282.698	4282.698	4282.698

3.2 Discussion

By referring to Table 4, some of the ID non-symmetric square matrices find the optimal solution for primal and dual problems by providing same objective function value in both softwares whereas in some ID non-symmetric square matrices, duality gap exists. Therefore, those matrices cannot obtain the optimal solution.

Based on the Duality theorem, for any couple of dual problem, the LP solution can either be optimal, infeasible or unbounded solution. Strong Duality stated that if solution couple (x, y) of the two problems has the feature that $f(x) = v(y)$, then x is said to be the optimal solution of primal LP problem and y is the best solution of dual LP problem. For weak duality, if primal LP problem does not have any finite optimization, then dual LP problem does not have any acceptable solutions and vice versa [18].

All matrices generated in this study were non-singular matrix and it can be called as full-rank matrix [19]. All the matrices were invertible and each constraint represented by the row in matrix was independent to each other [20]. In this experiment, some matrices could achieve optimal solution. Hence, it can be concluded that non-singular matrices do not guarantee the optimality of the duality LP solutions. Another finding in this paper is identifying the patterns of the ID random non-symmetric matrix that guarantee optimal solution and the features can be attained. Table 5 shows the characteristics of ID random non-symmetric square matrix.

Table 5: Characteristics of ID random non-symmetric square matrix

Criteria	Pattern
Diagonal entries	i. First diagonal entry is positive and remaining diagonal entries are negative. ii. All positive off-diagonal entries.
Leading principal minors	i. Negative value at the even orders and positive value at the odd orders of leading principal minor.

The first diagonal entry should be positive and the remaining diagonal entries should be negative to allow the generated matrices which consisted of the alternate sign of leading principal minors. The entries beside diagonal or known as off-diagonal entries must be positive. Random sign of off-diagonal entries can influence the value of determinants of leading principal minors. These findings allow the researchers to gain an in-depth understanding of the characteristics of ID random non-symmetric square matrices that could provide optimal solution.

4.0 CONCLUSION

Based on simulations results and validation of the primal-dual LP solutions, it can be concluded that ID square matrices could provide optimal solution to the LP problems or it may present duality gap and no optimal solution is revealed. Symmetric and non-symmetric ID random square matrices provide similar LP solutions. Although all generated ID non-symmetric square matrices are non-singular matrix, but only certain matrices could achieve optimal solution. Hence, the non-singular matrix does not guarantee the optimality of LP solutions.

For those ID non-symmetric square matrices that achieve optimal solution, these solutions agree with the theory of strong duality. If the ID non-symmetric square matrices with duality gap between the primal and the dual solutions or no optimal solution exists, the dual solutions are infeasible and these duality fit to the weak duality theorem. An ID square matrix can accomplish optimal solution to the LP problems if the first diagonal entry of the matrix is positive and the remaining diagonal entries are negative, with all positive off-diagonal entries. The determinants of the leading principal minors consist of negative value at even orders and positive value at odd orders which fulfill the condition of the ID matrix.

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