ON BUCKLING OF CYLINDRICAL SHELLS UNDER COMBINED LOADING

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ABSTRACT: This paper presented an analytical investigation of the buckling behavior of cylindrical shells under a combined action of thermal and mechanical loadings based on the general form of Green's strain tensor in curvilinear coordinates. While the shell was subjected to lateral pressure, it was assumed to be under either a uniform temperature increase or a uniform temperature gradient. A dimensionless load interaction parameter was considered to express the ratio of thermal and mechanical loads. The system of governing equations was derived using the harmonic series and was optimized with respect to harmonic numbers to find the critical buckling loads of the cylindrical shells. Results were calculated for both the Donnell and Green-types of kinematic nonlinearity. Comparison studies showed that both types of kinematic nonlinearity predicted the same critical buckling loads for thin cylindrical shells whereas for moderately thick cylindrical shells, the latter type of kinematic nonlinearity predicted higher critical buckling loads than the former type of kinematic nonlinearity.

KEYWORDS: *Green Strain Tensor; Buckling; Cylindrical Shell; Combined Loading*

1.0 INTRODUCTION

Cylindrical shells are important structural elements in marine vessels, aerospace structures and piping systems. Buckling characteristics of these structural elements are an extremely important factor in their design process. Various theoretical and numerical techniques have been proposed to contribute to this challenging task. Earlier works on the stability of cylindrical shells were carried out by Donnell in 1934 [1-2]. He presented a simple formula for critical buckling loads of isotropic cylindrical shells under axial compression [1] and torsion [2]. Later on, by introducing extra additional terms in kinematic relations, modified forms of Donnell shell theory were introduced [3-5]. Shear deformation shell theories were then developed for thick-walled shells by accounting transverse shear stresses in the shell theory [6-7]. An overview of these activities is given in [8-11]. Nevertheless, as stated in [12] and [13], in many studies on the buckling of shell structures, the loading condition has been either thermal or mechanical [14-16] and less attention has been paid to the buckling of shell structures under a combined action of thermal and mechanical loadings [13,17-19].

This study investigated effect of Green-type kinematic nonlinearity on critical buckling loads of thin and moderately thick cylindrical shells under combined lateral pressure and thermal loading. The displacement field was based on the first order shear deformation shell theory including the shear correction factor. Buckling loads are obtained for a given load interaction parameter which expresses the ratio of thermal and mechanical loads [5,19], thereby reducing double parameter stability equations to a single parameter equation. Results were calculated for both the Donnell and Green-types of kinematic nonlinearity.

2.0 METHODOLOGY

For a cylindrical shell of mean radius R, finite length L, and thickness h, the Green's strain tensor in cylindrical coordinates (x, θ, z) is [20].

$$\begin{aligned}
\epsilon^0_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right] \\
\epsilon^0_{\theta\theta} &= \frac{1}{R}\left(\frac{\partial v}{\partial \theta} + w\right) + \frac{1}{2R^2}\left[\left(\frac{\partial u}{\partial \theta}\right)^2 + \left(\frac{\partial w}{\partial \theta} - v\right)^2 + \left(\frac{\partial v}{\partial \theta} + w\right)^2\right] \\
\gamma^0_{x\theta} &= \left(\frac{\partial u}{R\partial \theta} + \frac{\partial v}{\partial x}\right) + \frac{1}{R}\left[\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial \theta}\right) + \frac{\partial v}{\partial x}\left(\frac{\partial v}{\partial \theta} + w\right) + \frac{\partial w}{\partial x}\left(\frac{\partial w}{\partial \theta} - v\right)\right]
\end{aligned} \tag{1}$$

where u, v and w are the axial, circumferential and lateral displacements of the cylindrical shell. Based on the Donnell's hypothesis [1-2], nonlinear terms are dependent on axial and circumferential displacements which can be neglected, thus we obtain:

$$\varepsilon_{xx}^{0}=\frac{\partial u}{\partial x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^{2},\ \varepsilon_{\theta\theta}^{0}=\frac{1}{R}\left(\frac{\partial v}{\partial \theta}+w\right)+\frac{1}{2R^{2}}\left(\frac{\partial w}{\partial \theta}\right)^{2},\ \gamma_{x\theta}^{0}=\left(\frac{\partial u}{R\partial \theta}+\frac{\partial v}{\partial x}\right)+\frac{1}{R}\left(\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta}\right) \tag{2}$$

Based on the first order shear deformation shell theory, the normal and shear strains of an arbitrary point of the cylindrical shell from its middle surface were given by the following relations [19]:

$$\begin{aligned}
&\varepsilon_{xx}=\varepsilon_{xx}^{0}+z\frac{\partial \phi_x}{\partial x}, \qquad \varepsilon_{\theta\theta}=\varepsilon_{\theta\theta}^{0}+z\frac{\partial \phi_\theta}{R\partial \theta}, \qquad \gamma_{x\theta}=\gamma_{x\theta}^{0}+z\left(\frac{\partial \phi_x}{R\partial \theta}+\frac{\partial \phi_\theta}{\partial x}\right)\\
&\gamma_{xz}=\phi_x+\frac{\partial w}{\partial x}, \qquad \gamma_{\theta z}=\phi_\theta+\frac{\partial w}{R\partial \theta}
\end{aligned} \tag{3}$$

The shear correction factor was set equal to 5/6 [21]. Using the variational approach [5], the equilibrium equations of cylindrical shells were

$$\begin{aligned}
&\frac{\partial N_{xx}}{\partial x}+\frac{\partial N_{x\theta}}{R\partial \theta}+N_{xx}\frac{\partial^2 u}{\partial x^2}+\frac{N_{\theta\theta}}{R^2}\frac{\partial^2 u}{\partial \theta^2}=0\\
&\frac{\partial N_{x\theta}}{\partial x}+\frac{\partial N_{\theta\theta}}{R\partial \theta}+N_{xx}\frac{\partial^2 v}{\partial x^2}+\frac{N_{\theta\theta}}{R^2}\left(\frac{\partial^2 v}{\partial \theta^2}+\frac{2\partial w}{\partial \theta}-v\right)=0\\
&\frac{\partial M_{xx}}{\partial x}+\frac{\partial M_{x\theta}}{R\partial \theta}-Q_{xz}=0\\
&\frac{\partial M_{x\theta}}{\partial x}+\frac{\partial M_{\theta\theta}}{R\partial \theta}-Q_{\theta z}=0\\
&\frac{Q_{xz}}{\partial x}+\frac{\partial Q_{\theta z}}{R\partial \theta}-\frac{N_{\theta\theta}}{R}+N_{xx}\frac{\partial^2 w}{\partial x^2}+\frac{N_{\theta\theta}}{R^2}\left(\frac{\partial^2 w}{\partial \theta^2}-w-\frac{2\partial v}{\partial \theta}\right)=0
\end{aligned} \tag{4}$$

where N_{ij} and M_{ij} (i, j=x, θ) were the stress and moment resultants and Q_{xz} and $Q_{\theta z}$ were the shear stress resultants. According to the adjacent equilibrium criterion [5], for an externally loaded cylindrical shell, the total displacements of a neighboring state of stability can be summed as the displacement components of equilibrium state and a neighboring stable state with respect to the equilibrium position as follows

$$u=u^{s}+u^{e},\ \ v=v^{s}+v^{e},\ \ w=w^{s}+w^{e},\ \ \phi_x=\phi_x^{s}+\phi_x^{e},\ \ \phi_\theta=\phi_\theta^{s}+\phi_\theta^{e} \tag{5}$$

where the superscript s describes the state of stability while the superscript e describes the state of equilibrium conditions. For simply supported cylindrical shells, the following expressions satisfy the circumferential periodicity

$$\begin{Bmatrix} (u^s, \phi_x^s) \\ (v^s, \phi_\theta^s) \\ w^s \end{Bmatrix} = \sum_m^\infty \sum_n^\infty \begin{Bmatrix} (U_{mn}, X_{mn})\cos(\bar{m}x)\cos(n\theta) \\ (V_{mn}, Y_{mn})\sin(\bar{m}x)\sin(n\theta) \\ W_{mn}\sin(\bar{m}x)\cos(n\theta) \end{Bmatrix} \tag{6}$$

where $\bar{m} = m\pi / L$, m is the half wave length in the x-direction, n is the wave number in the θ-direction, and U_{mn}, V_{mn}, W_{mn}, X_{mn} and Y_{mn} are undetermined coefficients. Using Equations (4)-(6) the stability equations of cylindrical shells under combined loads are derived as

$$U_{mn}\left[-\bar{m}^2 - \frac{1-\mu}{2R^2}n^2 - \frac{1-\mu^2}{Eh}\left(N_{xx}^e\bar{m}^2 + \frac{N_{\theta\theta}^e}{R^2}n^2\right)\right] + V_{mn}\left(\frac{\mu}{R}\bar{m}n + \frac{1-\mu}{2R}\bar{m}n\right) + W_{mn}\left(\frac{\mu}{R}\bar{m}\right) = 0 \tag{7a}$$

$$U_{mn}\left(\frac{\mu}{R}\bar{m}n + \frac{1-\mu}{2R}\bar{m}n\right) + V_{mn}\left[-\frac{n^2}{R^2} - \frac{1-\mu}{2}\bar{m}^2 - \frac{1-\mu^2}{Eh}\left(N_{xx}^e\bar{m}^2 + \frac{N_{\theta\theta}^e}{R^2}n^2 + \frac{N_{\theta\theta}^e}{R^2}\right)\right] + W_{mn}\left(-\frac{n}{R^2} - N_{\theta\theta}^e\frac{2n(1-\mu^2)}{EhR^2}\right) = 0 \tag{7b}$$

$$U_{mn}\left(\frac{\mu}{R}\bar{m}\right) + V_{mn}\left(-\frac{n}{R^2} - \frac{2N_{\theta\theta}^e}{R^2}n\right) + W_{mn}\left[-\frac{1-\mu}{2}\kappa\bar{m}^2 - \frac{1-\mu}{2R^2}\kappa n^2 - \frac{1}{R^2} - \frac{1-\mu^2}{Eh}\left(N_{xx}^e\bar{m}^2 + \frac{N_{\theta\theta}^e}{R^2}n^2 + \frac{N_{\theta\theta}^e}{R^2}\right)\right] + X_{mn}\left(-\frac{1-\mu}{2}\kappa\bar{m}\right) + Y_{mn}\left(\frac{1-\mu}{2R}\kappa n\right) = 0 \tag{7c}$$

$$W_{mn}\left(-\frac{6\kappa(1-\mu)}{h^2}\bar{m}\right) + X_{mn}\left(-\bar{m}^2 - \frac{1-\mu}{2R^2}n^2 - \frac{6\kappa(1-\mu)}{h^2}\right) + Y_{mn}\left(\frac{\mu}{R}\bar{m}n + \frac{1-\mu}{2R}\bar{m}n\right) = 0 \tag{7d}$$

$$W_{mn}\left(\frac{6\kappa(1-\mu)}{Rh^2}n\right) + X_{mn}\left(\frac{\mu}{R}\bar{m}n + \frac{1-\mu}{2R}\bar{m}n\right) + Y_{mn}\left(-\frac{n^2}{R^2} - \frac{1-\mu}{2}\bar{m}^2 - \frac{6\kappa(1-\mu)}{h^2}\right) = 0 \tag{7e}$$

where μ is the Poisson's ratio that is assumed to be 0.3 [21]. The equilibrium terms in Equations (7a-7e) satisfy the equilibrium condition and therefore drop out of the equations. Also, the nonlinear terms of equilibrium state are ignored because they are small compared to the other terms. The algebraic set of Equations (7a-7e) can be written in matrix form. By setting the determinant of the matrix to zero, the resulting equation could be optimized to find the critical buckling loads.

Two different thermal loadings are considered in this study, namely uniform temperature increase and uniform temperature gradient. The pre-buckling axial force for the uniform temperature gradient across the shell thickness is $N_{xx}^{e} = -E\alpha h^{2} T_{,z} / 2(1-\mu)$ with $T_{,z} = \Delta T / h$ and for the uniform temperature increase is $N_{xx}^{e} = -E\alpha h \Delta T / (1-\mu)$ [22]. Furthermore, the pre-buckling circumferential force for external pressure was given by $N_{\theta\theta}^{e} = -PR$. The non-dimensional load parameter η was defined to express the combination of the applied thermal loading and lateral pressure. That is: Case I: $N_{\theta\theta}^{e} = \eta N_{xx}^{e}$ for thermal stability analysis or Case II: $N_{xx}^{e} = \eta N_{\theta\theta}^{e}$ for mechanical stability analysis. Upon substitution of either of these equations into the resulting equation from the stability Equations (7a-7e), an equation with single load parameter was derived. An eigenvalue analysis is then carried out to obtain the critical buckling loads of cylindrical shells under combined loads.

3.0 RESULTS AND DISCUSSIONS

To demonstrate the efficiency of the proposed methodology, several comparison studies were presented for thermal and mechanical buckling of cylindrical shells. Table 1 compares the non-dimensional critical temperature ($\alpha\Delta T_{cr}$) for the uniform temperature increase and uniform temperature gradient when an isotopic cylindrical shell is subjected to a pure thermal loading. It can be seen from the table that for thin cylindrical shells , the critical temperatures obtained by the present study (from both types of kinematic nonlinearity) as a result of pure thermoelastic stability analysis are in excellent agreement with those obtained according to the Sander's assumptions [22]. However, for moderately thick cylindrical shells, the Green's strain tensor predicts smaller values for critical buckling loads of the cylindrical shells compared to the results obtained by the other two hypotheses. It is worth mentioning that the present calculations are based on the first order definition of strains while those reported in [22] are based on the classical kinematic relations.

For pure mechanical loading condition, the buckling pressure (kPa) of a Si_3N_4 cylindrical shell under internal pressure reported in [23] were taken into consideration. As shown in Table 2, for very thin cylindrical shells excellent agreement can be seen between the results obtained by the present study and those reported in [23], but for R/h=40, higher values of buckling pressure are produced by the Green's hypothesis.

Numerical calculations in [23] are obtained based on the third order shear deformation shell theory with von Kármán- Donnell type of kinematic nonlinearity. The effect of the load interaction parameter is shown in Tables 3 and 4. The numbers in brackets indicated the harmonic numbers. As noted in previous section, the critical loads were computed for two cases of load interaction parameter. It is found that when a combined load was applied to the cylindrical shell and the response of the shell was governed by the mechanical load, the variation of critical buckling loads over the shell thickness was followed by a marked decrease (note the axial load expressed in terms of pressure loading). Furthermore, the smallest value of m should be set to unit value while n should be determined by optimization.

Table 1: Non-dimensional critical temperature ($\alpha\Delta T_{cr}$) for cylindrical shells under pure thermal loading

Type of thermal loading	R/h	[22]	Present study	
			Donnell's hypothesis	Green's tensor
Uniform temperature increase	5	0.0848	0.0771	0.0756
	10	0.0424	0.0404	0.0401
	20	0.0212	0.0207	0.0206
	100	0.0042	0.0042	0.0042
	200	0.0021	0.0021	0.0021
	1000	0.0004	0.0004	0.0004
Uniform temperature gradient	5	0.1696	0.1543	0.1512
	10	0.0848	0.0808	0.0803
	20	0.0424	0.0414	0.0413
	100	0.0084	0.0084	0.0084
	200	0.0042	0.0042	0.0042
	1000	0.0008	0.0008	0.0008

Table 2: Critical pressure for cylindrical shells under pure mechanical loading (L^2/Rh=500)

$(m,n)_{cr}$	R/h	Source	Critical Pressure (kPa)
(1,4)	40	[23]	9112.24
		Present study (Donnell's hypothesis)	9112.17
		Present study (Green's tensor)	10351.93
(1,11)	400	[23]	87.4899
		Present study (Donnell's hypothesis)	87.4894
		Present study (Green's tensor)	88.9353

Table 3: Non-dimensional critical temperatures ($\alpha \Delta T_{cr} \times 10^3$) for a cylindrical shell (L/R=2) under combined load (Case I)

Type of thermal loading	R/h	η	Present study	
			Donnell's hypothesis	Green's tensor
Uniform temperature increase	5	0.5	45.161 (1,2)	47.373 (1,2)
		1	28.823 (1.3)	33.127 (1,3)
	10	0.5	13.366 (1,3)	17.914 (1,3)
		1	9.944 (1,3)	11.443 (1,3)
	20	0.5	6.347 (1,4)	6.853 (1,4)
		1	3.598 (1,4)	3.966 (1,4)
	50	0.5	1.666 (1,5)	1.767 (1,5)
		1	0.908 (1,5)	0.972 (1,5)
	100	0.5	0.604 (1,6)	0.631 (1,6)
		1	0.321 (1,6)	0.338 (1,6)
Uniform temperature gradient	5	0.5	90.323 (1,2)	94.747 (1,2)
		1	57.645 (1,3)	66.254 (1,3)
	10	0.5	32.732 (1,3)	35.827 (1,3)
		1	19.887 (1,3)	22.886 (1,3)
	20	0.5	12.694 (1,4)	13.706 (1,4)
		1	7.195 (1,4)	7.932 (1,4)
	50	0.5	3.332 (1,5)	3.534 (1,5)
		1	1.8154 (1,5)	1.944 (1,5)
	100	0.5	1.207 (1,6)	1.263 (1,6)
		1	0.642 (1,6)	0.675 (1,6)

Table 4: Non-dimensional critical pressure ($P_{cr}/E \times 10^3$) for a cylindrical shell (L/R=2) under combined load (Case II)

R/h	η	Present study	
		Donnell's hypothesis	Green's tensor
5	0.5	9.228 (1,3)	10.977 (1,3)
	1	8.235 (1,3)	9.465 (1,3)
10	0.5	1.591 (1,3)	1.898 (1,3)
	1	1.420 (1,3)	1.634 (1,3)
20	0.5	0.275 (1,4)	0.307 (1,4)
	1	0.257 (1,4)	0.283 (1,4)
50	0.5	0.027 (1,5)	0.029 (1,5)
	1	0.026 (1,5)	0.027 (1,5)
100	0.5	0.005 (1,6)	0.005 (1,6)
	1	0.005 (1,6)	0.005 (1,6)

4.0 CONCLUSIONS

A closed-from solution is presented to obtain the critical temperatures and pressures for thin and moderately thick cylindrical shells under combined thermal and mechanical loads. The stability equations are established using the Green's strain tensor in cylindrical coordinates. The analysis is carried out for two types of thermal loading including uniform temperature increase and temperature gradient. Both the Donnell's hypothesis and Green's strain tensor predict the same critical buckling loads for thin cylindrical shells but for moderately thick cylindrical shells, the Green's strain tensor produces higher critical buckling loads than the Donnell's hypothesis.

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